

# Strengthening constraints on Yukawa-type corrections to Newtonian gravity from measuring the Casimir force between a cylinder and a plate

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## Abstract

We discuss possibility to obtain stronger constraints on non-Newtonian gravity from measuring the gradient of the Casimir force between a cylinder and a plate. Exact analytical expression for the Yukawa-type force in a cylinder-plate configuration is obtained, as well as its asymptotic expansions. The gravitational force is compared with the Casimir force acting between a cylinder and a plate. Numerical computations for the prospective constraints on non-Newtonian gravity are performed for recently proposed experiment using a microfabricated cylinder attached to a micromachined oscillator. Specifically, it is shown that this experiment is expected to obtain up to 70 times stronger constraints on the Yukawa-type force, compared with the best present day limits, over a wide interaction range from 12.5 to 630 nm.

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## I. INTRODUCTION

The Yukawa-type corrections to Newtonian gravitational law have been discussed for a long time in connection with the problem of the so-called *fifth force* [1]. In the interaction of two atoms belonging to different macrobodies such corrections may arise due to the exchange of light hypothetical elementary particles predicted in many extensions of the Standard Model (scalar axion [2], graviphoton [3], dilaton [4], moduli [5] etc.). After the summation of atom-atom Yukawa-type interactions over the volumes of two macrobodies, one arrives at some force in addition to Newtonian gravity.

Interest in this kind of forces was rekindled after the proposal of extra-dimensional models for which the compactification energy can be as low as 1 TeV [6–8]. The respective characteristic size of the compact manifold was shown to be 1 mm and 5 nm for two and three compact extra dimensions, respectively [8]. At separations several times larger than the compactification scale it was predicted [9, 10] that the standard Newton’s gravitational law acquires the Yukawa-type correction. Keeping in mind that at separations below  $10\,\mu\text{m}$  corrections to Newton’s law that are many orders of magnitude larger than gravitation are not excluded experimentally, the predicted correction has attracted considerable interest. Specifically, a lot of gravitational experiments were performed which imposed limits on the strength of Yukawa-type interaction within short interaction ranges (see, for instance, Refs. [11–15]). From these experiments, strongest limits were obtained for Yukawa interaction ranges larger than a few micrometers.

For interaction ranges of about one micrometer and less the Casimir force occupies the place of Newtonian gravity as a dominant background force. The Casimir force [16] originates from quantum fluctuations of the electromagnetic field and finds diverse applications in both fundamental physics and nanotechnology (see monographs [17, 18]). Numerous experiments on measuring the Casimir force between a sphere and a plate were performed during the last few years (see Ref. [19] for a review). The measure of agreement between the experimental data and theory was used to impose limits on the parameters of hypothetical Yukawa-type interaction [20–28]. Eventually, the previously known constraints with interaction ranges below one micrometer were strengthened up to 10000 times.

Recently, the experiment on measuring the lateral Casimir force between sinusoidally corrugated surfaces of a sphere and a plate [29, 30] was used to constrain the Yukawa-type

force in the interaction range below 14 nm. Here, the measure of agreement between the data and the exact theory based on the scattering approach with no fitting parameters resulted [31] in the strengthening up to a factor of  $2.4 \times 10^7$ . As opposed to Ref. [23], where the confidence level could not be determined due to several uncertainties inherent to relevant experiment [32] on measuring the Casimir force between two crossed cylinders, the constraints of Ref. [31] were obtained at a 95% confidence level. In Ref. [31] a few prospective experiments were also considered using the configurations of a plate and a sphere and two parallel plates where surfaces of the test bodies are smooth or covered with sinusoidal corrugations. It was shown that some of these experiments are of high promise for further strengthening of constraints on the parameters of Yukawa-type hypothetical interaction.

Further prospects in the strengthening of constraints on non-Newtonian gravity from the measurements of the Casimir force are connected with the use of alternative configurations. It was argued [33], that the measurement setups using a cylinder above a plate or two eccentric cylinders combine some advantages of the plate-plate and sphere-plate configurations. Reference [33] noted that cylindrical geometries lead to favorable conditions for the search of extra-dimensional forces in the micrometer range. In Ref. [34] the experimental setup with a relatively large metallic cylinder of 6.35 mm diameter separated from the metallic plate with a gap of more than  $1 \mu\text{m}$  width was considered. Using the metal coated cylindrical lens of even larger 24 mm diameter, serious difficulties have been found, however, in the electrostatic calibration of Casimir apparatus [35]. Specifically, it was shown that the residual potential difference between the grounded test bodies may become dependent on the separation between them, and the force-distance relation for the electric force deviates from the form predicted by electrodynamics in a cylinder-plate geometry. Similar anomalies were earlier found [36] in the sphere-plate geometry for spheres of centimeter-size curvature radii (for spheres of about  $100 \mu\text{m}$  radii anomalies do not appear [25–30, 37]). In Ref. [38] they were explained by unavoidable deviations of mechanically polished and grounded surfaces from perfect spherical shape assumed in calculations. As recognized in Ref. [35], deviations of the surface from perfect cylindrical shape might be also responsible for the calibration problems arising in the case of a centimeter-size cylinder above a plate. Because of this, in Ref. [39] the measurement of the Casimir force between a microfabricated cylinder attached to the micromechanical oscillator and a plate was proposed. The respective setup uses cylinders with radii of about  $100 \mu\text{m}$  and gets the most benefit from the high precision offered

by a micromachined oscillator.

In this paper we investigate prospective constraints on the parameters of Yukawa-type corrections to Newtonian gravity which can be obtained from the measurements of the Casimir force in the cylinder-plate geometry. The Yukawa-type force between a cylinder and a plate is calculated analytically both exactly and using the most general form of the proximity force approximation (PFA), i.e., the so-called *Derjaguin method* [40], with coinciding results. The respective expressions are obtained both for homogeneous test bodies and for bodies covered with layers made of different materials. This allows immediate application of the obtained results to the experiments in preparation. The gravitational force acting between a cylinder and a plate is calculated and compared with the Casimir force. The range of parameters is determined where the gravitational force can be discounted when obtaining constraints on the Yukawa-type corrections to Newtonian gravity from the measurements of the Casimir force between a cylinder and a plate. The plate-based and cylinder-based versions of the PFA [41, 42] are considered in application to the calculation of the gravitational force. Numerical computations are performed for the experiment using a microfabricated cylinder attached to a micromachined oscillator. Prospective constraints on the parameters of Yukawa-type corrections to Newtonian gravity that can be obtained from the measurement of the Casimir force between a plate and a microfabricated cylinder promise to be up to a factor of 70 stronger than the previously known ones found in Refs. [28, 31]. This provides further impetus to the planned experiments on measuring the Casimir force in cylinder-plane geometry. The obtained expressions for the Yukawa-type interaction can be used not only for a microfabricated cylinder, but in any other experiment exploiting the cylinder-plate geometry.

The paper is organized as follows. In Sec. II we calculate the Yukawa-type force in the configuration of a cylinder above a plate. Section III is devoted to the calculation of the gravitational force and to the comparison between gravitational and Casimir forces. In Sec. IV the prospective constraints on the hypothetical Yukawa-type interaction are derived. Section V contains our conclusions and discussion.

## II. YUKAWA-TYPE FORCE IN CYLINDER-PLATE CONFIGURATION

We consider a circular cylinder of radius  $R$ , density  $\rho_2$  and length  $L$  situated above a large (i.e., with a dimension much larger than  $L$  and  $2R$ ) plane plate of density  $\rho_1$  and thickness  $D_1$  parallel to it. Let the  $z$ -axis be perpendicular to the plate, the upper surface of the plate be the plane  $z = 0$  and the  $y$ -axis be parallel to the cylinder axis. The separation distance between the plate and the cylinder is denoted  $a$ .

The Yukawa interaction energy between an atom of mass  $m_2$  belonging to the cylinder and an atom of mass  $m_1$  at a separation  $r$  belonging to the plate is conventionally represented in the form

$$V_{\text{Yu}}(r) = -\frac{Gm_1m_2}{r} \alpha e^{-r/\lambda}, \quad (1)$$

where  $G$  is the Newtonian gravitational constant,  $\alpha$  is a dimensionless constant characterizing the strength of the Yukawa interaction, and  $\lambda$  is its interaction range. Let us suppose that the atom  $m_2$  is at a height  $z$  above the plate. Integration of Eq. (1) over the volume of the plate and subsequent negative differentiation with respect to  $z$  results in the Yukawa force between the atom  $m_2$  and the plate [43]

$$F_{\text{Yu}}^{(m_2,p)}(z) = -2\pi Gm_2\rho_1\alpha\lambda e^{-z/\lambda} (1 - e^{-D_1/\lambda}). \quad (2)$$

To obtain the Yukawa-type force between a plate and a cylinder, one should integrate Eq. (2) over the volume of the cylinder. This leads to

$$\begin{aligned} F_{\text{Yu}}^{(c,p)}(a) &= -4\pi G\rho_1\rho_2\alpha\lambda L (1 - e^{-D_1/\lambda}) \\ &\quad \times \int_a^{a+2R} dz e^{-z/\lambda} [R^2 - (R + a - z)^2]^{1/2} \\ &= -4\pi G\rho_1\rho_2\alpha\lambda L (1 - e^{-D_1/\lambda}) e^{-a/\lambda} \\ &\quad \times \int_0^{2R} dv e^{-v/\lambda} (2Rv - v^2)^{1/2}, \end{aligned} \quad (3)$$

where the new integration variable  $v = z - a$  was introduced. Integrating by parts and using formula 3.366.1 in Ref. [44], we arrive at the result

$$\begin{aligned} F_{\text{Yu}}^{(c,p)}(a) &= -4\pi^2 G\rho_1\rho_2\alpha\lambda^2 LR (1 - e^{-D_1/\lambda}) \\ &\quad \times e^{-(R+a)/\lambda} I_1(R/\lambda), \end{aligned} \quad (4)$$

where  $I_n(z)$  is the Bessel function of imaginary argument.

The interaction ranges of the Yukawa-type forces discussed below are  $\lambda \leq 1 \mu\text{m}$ , whereas cylinder radii are  $R \geq 50 \mu\text{m}$ . This means that  $R/\lambda \gg 1$  and one can use the asymptotic expansion of large arguments [45]

$$I_1(R/\lambda) = \sqrt{\frac{\lambda}{2\pi R}} e^{R/\lambda} \left( 1 + \tilde{u}_1 \frac{\lambda}{R} + \tilde{u}_2 \frac{\lambda^2}{R^2} + \cdots \right), \quad (5)$$

where  $\tilde{u}_1, \tilde{u}_2, \dots$  are some numbers. Substituting this in Eq. (4) and preserving only the main contribution, we obtain the following expression for the Yukawa-type force between a plate and a cylinder valid under the condition  $\lambda \ll R$ :

$$F_{\text{Yu}}^{(c,p)}(a) \approx -2\pi G \rho_1 \rho_2 \alpha \lambda^2 L \sqrt{2\pi R \lambda} e^{-a/\lambda} (1 - e^{-D_1/\lambda}). \quad (6)$$

It is interesting to compare the approximate result (6) with the exact result (4). Thus, at  $\lambda/R$  equal to 0.001, 0.003, 0.005, and 0.01, the relative difference between the Yukawa-type forces computed using Eqs. (4) and (6) is equal to 0.038%, 0.11%, 0.19%, and 0.38%, respectively.

Taking into account that the Yukawa force is of potential nature, its exact expression for any compact body above a plate can be obtained [43, 46] by using the general formulation of the PFA suggested by Derjaguin [40]. According to this formulation, the compact body is replaced with a set of partial plane plates of appropriate thickness  $D_2(x)$  parallel to the underlying plate. Then the Yukawa force can be calculated as

$$F_{\text{Yu, PFA}}(a) = \int_{\sigma} P_{\text{Yu}}(z) d\sigma, \quad (7)$$

where  $P_{\text{Yu}}(z)$  is the Yukawa pressure between two infinite plane parallel plates of thicknesses  $D_1$  and  $D_2$  at a separation  $z$ , and  $\sigma$  is the projection of a compact body on the underlying plate (i.e., on the plane  $z = 0$ ). Later, such an approach was called the *plate-based* PFA [41, 42]. The expression for  $P_{\text{Yu}}(z)$  can be simply obtained from Eq. (2) by integration over the volume of the upper plate of thickness  $D_2$ . It is given by [43]

$$P_{\text{Yu}}(z) = -2\pi G \rho_1 \rho_2 \alpha \lambda^2 e^{-z/\lambda} \times (1 - e^{-D_1/\lambda}) (1 - e^{-D_2/\lambda}), \quad (8)$$

where for a cylinder above a plate the thickness of a partial plate and its separation from the underlying plate are given by  $D_2(x) = 2\sqrt{R^2 - x^2}$ ,  $z(x) = a + R - \sqrt{R^2 - x^2}$ . Substituting Eq. (8) into Eq. (7) we obtain  $F_{\text{Yu, PFA}}^{(c,p)}(a) = F_{\text{Yu}}^{(c,p)}(a)$ , where  $F_{\text{Yu}}^{(c,p)}(a)$  is defined in Eq. (4),

i.e., the Yukawa force calculated by means of the plate-based PFA coincides with the exact result, as it should be.

In experiments on measuring the Casimir force [18, 19] the test bodies of densities  $\rho_1$  and  $\rho_2$  are usually coated with two additional metallic layers. Let the plate be coated with a layer of density  $\rho'_1$  and thickness  $\Delta'_1$  and with an outer layer of density  $\rho''_1$  and thickness  $\Delta''_1$ . In so doing  $D_1$  is the plate thickness including layers. In a similar way, we assume that the cylinder is coated with a layer of density  $\rho'_2$  and thickness  $\Delta'_2$  and with an outer layer of density  $\rho''_2$  and thickness  $\Delta''_2$ . The radius of the cylinder coated with two layers is  $R$ . Equation (4) for the Yukawa-type force can be applied to a plate of thickness  $\Delta''_1$  and density  $\rho''_1$  interacting with a cylinder of radius  $R - \Delta'_2 - \Delta''_2$  and density  $\rho_2$  at a separation  $a + \Delta'_2 + \Delta''_2$  from the plate. Then one should take into account the interaction of the same plate  $\Delta''_1$ ,  $\rho''_1$  with a cylindrical layer of thickness  $\Delta'_2$  and density  $\rho'_2$ . This is achieved by applying Eq. (4) twice: to a cylinder of radius  $R - \Delta''_2$  and density  $\rho'_2$  at a separation  $a + \Delta''_2$  and to a cylinder of radius  $R - \Delta'_2 - \Delta''_2$  of the same density at a separation  $a + \Delta'_2 + \Delta''_2$ . The obtained results should be subtracted. Using this procedure, which is based on the additivity of Yukawa-type force for a necessary number of times, one can simply determine the contribution to the force of each covering layer. Performing summation of all contributions, we arrive at the exact Yukawa-type force acting between a plate and a cylinder coated with two layers each:

$$F_{\text{Yu},l}^{(c,p)}(a) = -4\pi^2 G\alpha\lambda^2 L e^{-a/\lambda} \Phi_p \Phi_c. \quad (9)$$

Here, the following notations are introduced:

$$\begin{aligned} \Phi_p &= \left[ \rho''_1 - (\rho''_1 - \rho'_1) e^{-\Delta''_1/\lambda} \right. \\ &\quad \left. - (\rho'_1 - \rho_1) e^{-(\Delta'_1 + \Delta''_1)/\lambda} - \rho_1 e^{-D_1/\lambda} \right], \\ \Phi_c &= e^{-R/\lambda} \left[ R\rho''_2 I_1 \left( \frac{R}{\lambda} \right) \right. \\ &\quad - (\rho''_2 - \rho'_2)(R - \Delta''_2) I_1 \left( \frac{R - \Delta''_2}{\lambda} \right) \\ &\quad \left. - (\rho'_2 - \rho_2)(R - \Delta'_2 - \Delta''_2) I_1 \left( \frac{R - \Delta'_2 - \Delta''_2}{\lambda} \right) \right]. \end{aligned} \quad (10)$$

As was noted above, the interaction range  $\lambda$  under consideration is much less than the cylinder radius,  $\lambda \ll R$ . Taking into account that the covering layers are sufficiently thin, i.e., the inequalities  $R - \Delta''_2 \gg \lambda$  and  $R - \Delta'_2 - \Delta''_2 \gg \lambda$  are satisfied, the expression for  $\Phi_c$

in Eq. (10) can be simplified

$$\Phi_c \approx \sqrt{\frac{\lambda}{2\pi}} \left[ \sqrt{R\rho_2''} - \sqrt{R - \Delta_2''}(\rho_2'' - \rho_2') e^{-\Delta_2''/\lambda} - \sqrt{R - \Delta_2' - \Delta_2''}(\rho_2' - \rho_2) e^{-(\Delta_2 + \Delta_2'')/\lambda} \right]. \quad (11)$$

If, in addition, the conditions  $\Delta_2'' \ll R$  and  $\Delta_2' + \Delta_2'' \ll R$  hold (this is, for instance, the case for experiments on measuring the Casimir force in sphere-plate geometry), then further simplification of Eq. (11) leads to

$$\Phi_c \approx \sqrt{\frac{\lambda R}{2\pi}} \left[ \rho_2'' - (\rho_2'' - \rho_2') e^{-\Delta_2''/\lambda} - (\rho_2' - \rho_2) e^{-(\Delta_2 + \Delta_2'')/\lambda} \right]. \quad (12)$$

Equations (9)–(12) allow computation of the Yukawa-type correction to Newtonian gravity in the experimental configuration of a cylinder above a plate. They will be used in Sec. IV for the estimation of prospective constraints on  $\alpha$  and  $\lambda$  which can be obtained from the measurement of the Casimir force between a plate and a microfabricated cylinder.

### III. COMPARISON OF GRAVITATIONAL AND CASIMIR FORCES

As discussed in Sec. I, strongest constraints on the Yukawa-type interaction follow from measurements of the Casimir force at interaction ranges where this force is the dominant background force. Here, we compare the gravitational and Casimir forces in the configuration of a cylinder and a plate and find where the gravitational force is negligibly small.

The Newtonian potential of the gravitational force between atoms  $m_1$  and  $m_2$  belonging to the plate and the cylinder, respectively, is given by

$$V_G = -\frac{Gm_1m_2}{r} \quad (13)$$

As in Sec. II, we consider an atom  $m_2$  at a height  $z$  above the plate. The gravitational force between an atom  $m_2$  and a plate is obtained [43] by the integration of Eq. (13) over the volume of the plate and subsequent negative differentiation with respect to  $z$

$$F_G^{(m_2,p)}(z) = -2\pi G\rho_1 m_2 D_1. \quad (14)$$

It is seen that the gravitational force does not depend on  $z$ . The gravitational force between a plate and a cylinder is obtained from the integration of Eq. (14) over the cylinder volume

$$F_G^{(c,p)} = -2\pi^2 G\rho_1 \rho_2 D_1 L R^2. \quad (15)$$



The same result is obtainable with the help of the plate-based PFA. Considering the second plate of thickness  $D_2$  and density  $\rho_2$  parallel to the first, the gravitational pressure between the two plates can be obtained [43] using Eq. (14)

$$P_G = -2\pi G\rho_1\rho_2 D_1 D_2. \quad (16)$$

Then the PFA result for the gravitational force between a plate and a cylinder is found from Eq. (7), where  $P_{Yu}$  should be replaced with  $P_G$

$$\begin{aligned} F_{G, \text{PFA}}^{(c,p)} &= -4\pi G\rho_1\rho_2 D_1 L \int_0^R dx D_2(x) \\ &= -2\pi^2 G\rho_1\rho_2 D_1 L R^2. \end{aligned} \quad (17)$$

Here,  $D_2(x)$  was used as defined below Eq. (8). From Eq. (17) it can be seen that the PFA result obtained using the Derjaguin method coincides with the exact result (15). Note that the cylinder-based PFA does not lead to correct results for the gravitational force [specifically, for a cylinder above a plate a factor of two larger force magnitude is obtained using this method, as compared with Eq. (17)].

The Casimir force acting between a metal coated cylinder and a metal coated plate in thermal equilibrium at temperature  $T$  was calculated in Ref. [39] using the plate-based PFA. It is given by

$$\begin{aligned} F_C^{(c,p)}(a, T) &= -\frac{k_B T L}{4\sqrt{\pi} a^2} \sqrt{\frac{R}{2a}} \sum_{l=0}^{\infty}{}' \int_{\tau l}^{\infty} v^{3/2} dv \\ &\times [\text{Li}_{1/2}(r_{\text{TM}}^2 e^{-v}) + \text{Li}_{1/2}(r_{\text{TE}}^2 e^{-v})]. \end{aligned} \quad (18)$$

Here,  $k_B$  is the Boltzmann constant,  $\text{Li}_n(z)$  is the polylogarithm function,  $\tau = 4\pi k_B T a / (\hbar c)$ ,  $l = 0, 1, 2, \dots$ , and prime near the summation sign adds a multiple  $1/2$  to the term with  $l = 0$ . The reflection coefficients for the two independent polarizations of the electromagnetic field (transverse magnetic, TM, and transverse electric, TE) are given by

$$\begin{aligned} r_{\text{TM}} &= \frac{\varepsilon_l v - \sqrt{v^2 + (\varepsilon_l - 1)\tau^2 l^2}}{\varepsilon_l v + \sqrt{v^2 + (\varepsilon_l - 1)\tau^2 l^2}}, \\ r_{\text{TE}} &= \frac{v - \sqrt{v^2 + (\varepsilon_l - 1)\tau^2 l^2}}{v + \sqrt{v^2 + (\varepsilon_l - 1)\tau^2 l^2}}, \end{aligned} \quad (19)$$

where the dielectric permittivity  $\varepsilon_l = \varepsilon(i\xi_l)$  is calculated at the imaginary Matsubara frequencies  $\xi_l = \tau l c / (2a)$ . For Au coated bodies used in experiments on measuring the Casimir

force  $\varepsilon(\omega)$  was described by means of the experimentally consistent generalized plasma-like model [18, 19, 28, 39] with the plasma frequency  $\omega_p = 9.0 \text{ eV}$  [48] (note that if a plate and a cylinder are coated with metallic layers, only external layers of thicknesses  $\Delta'_1, \Delta'_2$  determine the value of the Casimir force). The use of some alternative model of dielectric permittivity of Au (for instance, on the basis of tabulated optical data extrapolated to low frequencies by means of the Drude model) leads to almost the same result with respect to the comparison between Casimir and gravitational forces at separations below a micrometer.

Now we present a few computational results for the gravitational force and for the ratio of gravitational to Casimir forces. First, we consider the plate of  $D_1 = 5 \text{ mm}$  thickness and cylinders of different radii made of bulk Au with the density  $19.28 \times 10^3 \text{ kg/m}^3$ . In this case, the values of the gravitational force per unit cylinder length calculated using Eq. (15) are equal to 6.12, 24.47, and 55.06 pN/m for cylinder radii 50, 100, and 150  $\mu\text{m}$ , respectively. The ratio  $F_G^{(c,p)}/F_C^{(c,p)}$  is equal to  $5.24 \times 10^{-4}$ ,  $1.48 \times 10^{-3}$ , and  $2.72 \times 10^{-3}$  with the same respective values of cylinder radii at the plate-cylinder separation distance  $a = 1 \mu\text{m}$ . This ratio increases with the increase of  $a$ . For instance, at  $a = 2 \mu\text{m}$  it is equal to  $5.29 \times 10^{-3}$ ,  $1.50 \times 10^{-2}$ , and  $2.75 \times 10^{-2}$  for cylinder radii 50, 100, and 150  $\mu\text{m}$ , respectively. In Fig. 1(a) we present the values of the ratio  $F_G^{(c,p)}/F_C^{(c,p)}$  as a function of separation for cylinder radii  $R = 50 \mu\text{m}$  (line 1), 100  $\mu\text{m}$  (line 2), and 150  $\mu\text{m}$  (line 3). It is still assumed that both the cylinder and the plate are made of bulk Au. As can be seen in Fig. 1(a), for microfabricated cylinders the gravitational force achieves some noticeable fraction of the Casimir force only at separations of a few micrometers.

In real experiments involving the layer structure of the test bodies the role of the gravitational force, as compared to the Casimir force, is even less. To illustrate this, we consider the generalization of Eq. (15) for the case of test bodies coated with two metal layers each (see Sec. II for all notations). Using the additivity of the gravitational force, the role of layers can be accounted for in the same way as it was done for the Yukawa-type force in Sec. II. As a result, Eq. (15) is replaced with

$$\begin{aligned}
F_{G,l}^{(c,p)} = & -2\pi^2 GL [\rho_1'' \Delta_1'' + \rho_1' \Delta_1' + \rho_1 (D_1 - \Delta_1' - \Delta_1'')] \\
& \times [\rho_2'' R^2 - (\rho_2'' - \rho_2')(R - \Delta_2'')^2 \\
& - (\rho_2' - \rho_2)(R - \Delta_2' - \Delta_2'')^2].
\end{aligned} \tag{20}$$

For numerical calculations we consider the same layer structure of the test bodies as in

Refs. [27, 28], i.e., Si plate coated with layers of Cr and Au and sapphire cylinder also coated with Cr and Au layers. The respective densities and thicknesses are the following:  $\rho_1'' = \rho_2'' = 19.28 \times 10^3 \text{ kg/m}^3$ ,  $\rho_1' = \rho_2' = 7.14 \times 10^3 \text{ kg/m}^3$ ,  $\rho_1 = 2.33 \times 10^3 \text{ kg/m}^3$ ,  $\rho_2 = 4.1 \times 10^3 \text{ kg/m}^3$ ,  $\Delta_1'' = 210 \text{ nm}$ ,  $\Delta_2'' = 180 \text{ nm}$ ,  $\Delta_1' = \Delta_2' = 10 \text{ nm}$ , and  $D_1 = 5 \text{ mm}$ . The computational results for the ratio  $F_{G,l}^{(c,p)}/F_C^{(c,p)}$  in the case of test bodies with layer structure are shown in Fig. 1(b) as a function of separation over the region from 1 to 5  $\mu\text{m}$  (below 1  $\mu\text{m}$  this ratio is vanishingly small). Lines 1, 2, and 3 are plotted for cylinder radii  $R = 50, 100$  and 150  $\mu\text{m}$ , respectively. As can be seen in Fig. 1(b), for a cylinder and a plate with layer structure used in real experiments on measuring the Casimir force the ratio of gravitational and Casimir force is much less than for bulk Au test bodies. Even at a separation  $a = 5 \mu\text{m}$  and for largest cylinder radius  $R = 150 \mu\text{m}$  this ratio is less than 0.01. This allows us to conclude that in the interaction range below 1  $\mu\text{m}$  under consideration in the next section the gravitational force between a plate and a microfabricated cylinder does not play any role and the Yukawa-type correction to gravity should be considered in the background of the Casimir force.

#### IV. PROSPECTIVE CONSTRAINTS

In this section we estimate the strength of constraints on the parameters of Yukawa-type corrections to Newtonian gravity that can be obtained from the measurements of the Casimir force between a microfabricated cylinder and a plate proposed in Ref. [39]. Similar to already performed experiments of Refs. [25–28], the experiment of Ref. [39] is dynamic, which is to say that the separation distance between the cylinder and the plate is varied harmonically at the natural frequency of the micromachined oscillator. The immediately measured quantity is the shift of this frequency under the influence of the Casimir force. In its turn, from the solution of a simple mechanical problem it follows that this shift of the resonance frequency is proportional to the gradient of the Casimir force [49, 50].

Below we assume that the confidence interval for the difference between theoretical and experimental gradients of the Casimir force in cylinder-plate configuration determined at a 95% confidence level  $[-\Theta(a), \Theta(a)]$  is the same as in Ref. [28]. Keeping in mind that the half width of the confidence interval  $\tilde{\Xi}(a)$  in Ref. [28] was determined for the difference of the Casimir pressure between two parallel plates, which is connected with the gradient of

the Casimir force in sphere-plate geometry by the equation

$$P_C(a) = -\frac{1}{2\pi R} \frac{\partial F_C^{(s,p)}(a)}{\partial a}, \quad (21)$$

we arrive at the equality  $\Theta(a) = 2\pi R \tilde{\Xi}(a)$ , where  $\tilde{\Xi}(a)$  at different separations is specified in Ref. [28]. A cylinder of  $R = 151.3 \mu\text{m}$  radius is used, i.e., one of the same radius as that of the sphere in Ref. [28]. The length of the cylinder  $L = \pi R/2$  is found from the condition that its projection on the plate is the same as the projection of the sphere.

Prospective constraints on the Yukawa-type corrections to Newtonian gravity can be estimated from the assumption that in the limit of the confidence interval  $\Theta(a)$  the hypothetical Yukawa interaction will not be observed. This leads to the inequality

$$\left| \frac{\partial F_{\text{Yu},l}^{(c,p)}(a)}{\partial a} \right| \leq \Theta(a), \quad (22)$$

where the gradient of the Yukawa force can be found from Eq. (9):

$$\frac{\partial F_{\text{Yu},l}^{(c,p)}(a)}{\partial a} = 4\pi^2 G \alpha \lambda L e^{-a/\lambda} \Phi_p \Phi_c. \quad (23)$$

The exact expressions for the factors  $\Phi_p$  and  $\Phi_c$  are presented in Eq. (10), and the asymptotic representations for  $\Phi_c$  in the case of large cylinder radii, thin layers and short interaction ranges in Eqs. (11) and (12).

The prospective constraints on  $\lambda$  and  $\alpha$  following from Eqs. (22) and (23) are shown in Fig. 2 by the dashed line, where the region of  $(\lambda, \alpha)$ -plane above the line is expected to be prohibited from the planned measurements of the gradient of the Casimir force between a cylinder and a plate and the region below this line is allowed. Note that strongest constraints shown in Fig. 2 are obtained at different separations  $a$ . Thus, in the region  $10 \text{ nm} \leq \lambda \leq 40 \text{ nm}$  the constraints shown by the dashed line are obtained at  $a = 180 \text{ nm}$ , in the region  $50 \text{ nm} \leq \lambda \leq 63 \text{ nm}$  at  $a = 200 \text{ nm}$  etc. For  $\lambda \leq 315 \text{ nm}$  the asymptotic expressions (11) and (12) for the factor  $\Phi_c$  lead to the same constraints as exact Eq. (10). As noted in Ref. [47], the half width of the confidence interval is determined not more precisely than with two or, at maximum, three significant figures. Because of this, independently of how precise the measurements of the Casimir force are, the Yukawa force never needs to be calculated with a higher than up to 0.1% accuracy.

For comparison purposes, in Fig. 2 we also plot the strongest constraints on the parameters of Yukawa-type interaction obtained up to now (solid lines 1–4). Line 1 is obtained

in Ref. [31] from measurements of the lateral Casimir force between corrugated surfaces [29, 30]. Line 2 follows from dynamic determination of the Casimir pressure by means of a micromechanical oscillator [28, 47]. Line 3 is plotted using the results of Casimir-less experiment [51] and line 4 is obtained [20] from the results of torsion pendulum experiment [52]. As can be seen in Fig. 2, the strongest strengthening of presently known constraints up to 70 times can be achieved at  $\lambda = 18$  nm. At  $\lambda = 80$  and 422 nm the expected strengthening of constraints is by a factor 28 and 7, respectively. All of all, the proposed experiment promises to strengthen the presently known constraints over the very wide interaction range  $12.5 \text{ nm} \leq \lambda \leq 630 \text{ nm}$ .

## V. CONCLUSIONS AND DISCUSSION

In the foregoing we have investigated the potentialities of experiments on measuring the Casimir force in cylinder-plate geometry for strengthening constraints on Yukawa-type corrections to Newtonian gravity. The configuration of a cylinder above a plate has long attracted attention with respect to measurements of the Casimir force [33–35, 53, 54]. Interest to it has rekindled after the suggestion [39] to use microfabricated cylinders and the setup of a micromachined oscillator which allowed to achieve highest experimental precision for the configuration of a sphere and a plate.

Here, we have obtained exact analytical expression for the Yukawa-type force acting between a cylinder and a large plate. This expression is generalized for the case of test bodies coated with thin metallic layers, as is the case in typical experiments on measuring the Casimir force and its gradient. It was shown that the same expression can be obtained using the Derjaguin formulation of the PFA. The magnitude of the gravitational force acting between a cylinder and a plate was compared with the magnitude of the Casimir force, and the former was shown to be negligibly small at the separations under consideration. With respect to the gravitational force, the plate-based and the cylinder-based versions of the PFA were discussed and found not to be equivalent. Starting from the assumption that main parameters of the proposed experiment on the measurement of the gradient of the Casimir force between a cylinder and a plate are the same as in the experiment of Ref. [28], we have estimated the prospective constraints on the Yukawa-type interaction obtainable using a microfabricated cylinder attached to a micromachined oscillator. It was shown that

the strengthening of constraints up to a factor 70 is expected over the wide interaction range from  $\lambda = 12.5$  nm to  $\lambda = 630$  nm. The obtained expressions for the Yukawa interaction can be used in all experiments exploiting the cylinder-plate geometry. At the moment, the strongest constraints valid within this interaction region are obtained from four different experiments on measuring the Casimir force.

The presented above results show that the Casimir effect has potentialities as the source of stronger constraints on non-Newtonian gravity. While the cylinder-plate configuration considered here promises obtaining stronger constraints in the interaction range below  $1\text{ }\mu\text{m}$ , it might be expected that the proposed experiment with two parallel plates coated with gold [55] will result in stronger constraints on the Yukawa force in the interaction range of a few micrometers.

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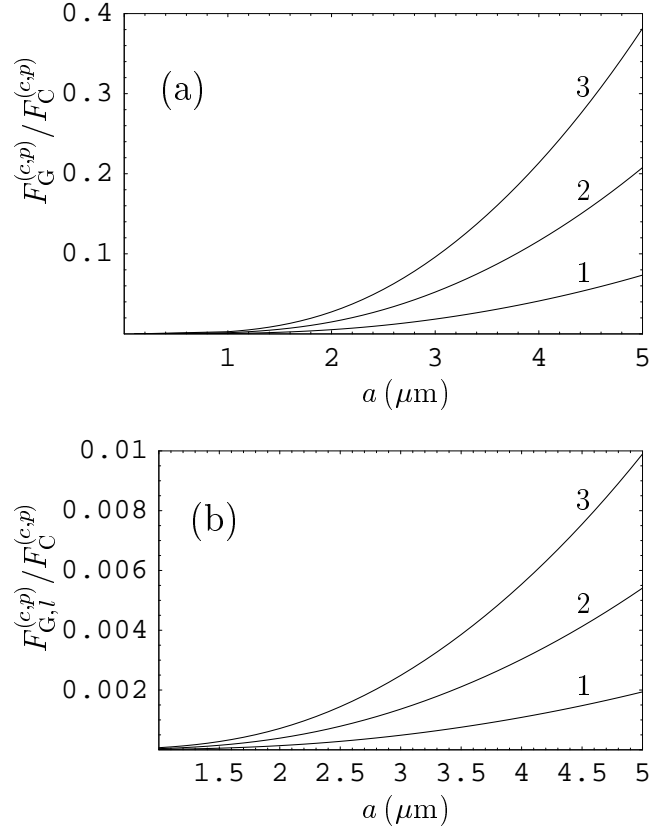


FIG. 1: Ratio between the gravitational and Casimir forces in a cylinder-plate configuration (a) for bulk gold bodies and (b) for bodies coated with thin metallic layers (see text for further details). The thickness of the plate is 5 mm. The computational results for cylinder radii  $R = 50, 100$ , and  $150 \mu\text{m}$  are shown by the lines 1, 2, and 3, respectively.

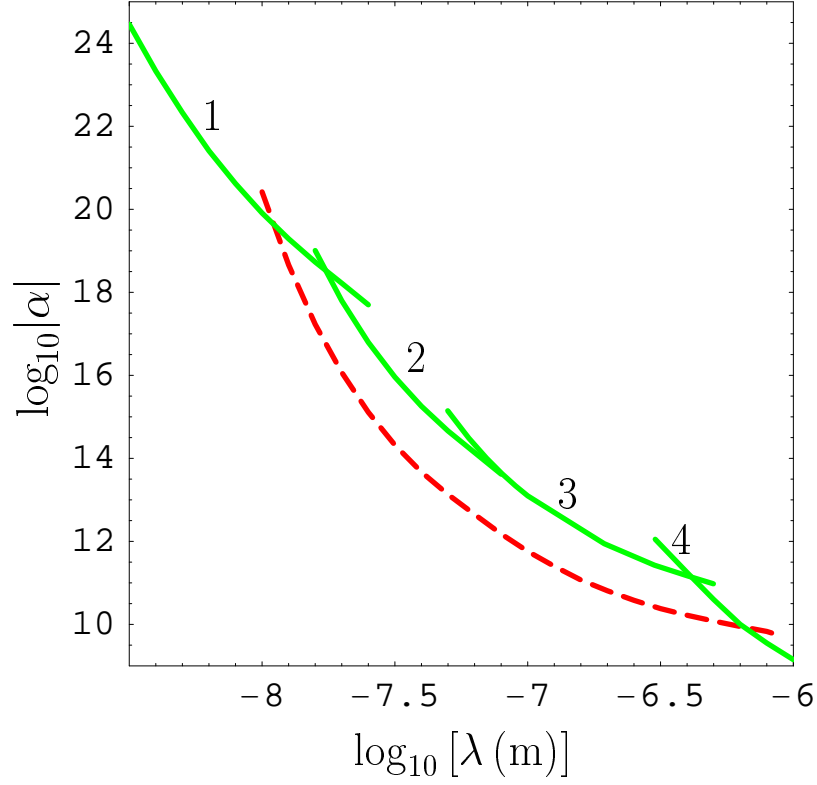


FIG. 2: (Color online) The strongest currently available constraints on the parameters of Yukawa-type force are shown by the solid lines 1–4 (see text for further discussion). The dashed line shows the prospective constraints that can be obtained from measuring the gradient of the Casimir force between a plate and a microfabricated cylinder. The allowed regions in the  $(\lambda, \alpha)$ -plane lie beneath the lines.